JOURNAL OF NUMBER THEORY 5, 203–214 (1973)

On Character Sums and Class Numbers

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Communicated by H. Halberstam

Received March 27, 1971; revised August 21, 1971

A mean-value estimate for character sums with real characters is proved in an elementary way. The application is a proof of a conjecture on the error term in the asymptotic expression of the sum $\sum_{d\leqslant X} h^k(-d)$, where $h(-d)$ is the class number of the quadratic field with the negative discriminant $-d$, and k is an integer $\geqslant 2$.

INTRODUCTION

1. All real primitive characters are given by the Kronecker symbol $\chi^{(d)}(n) = \left(\frac{d}{n}\right)$, where d is a fundamental discriminant or 1, i.e., either d square-free and $d \equiv 1 \pmod 4$, or $d = 4N$, N square-free and $N \equiv 2$ or $3 \pmod 4$. The modulus of $\chi^{(d)}$ is equal to $|d|$. Any real character is equivalent to a unique real primitive character and is, therefore, of the form $\left(\frac{da^2}{n}\right)$, where d is a fundamental discriminant or 1, and $(a, d) = 1$, a square-free. Hence, all real nonprincipal characters are of the form $\left(\frac{D}{n}\right)$, with D belonging to a set $\mathscr{D}$, defined from the following conditions:

(i) D is no square,

(ii) $D \equiv 1 \pmod 4$, or $D = 4N$, $N \equiv 1, 2,$ or $3 \pmod 4$.

The object of this paper is to investigate sums of the shape

$$S(X, Y) = \sum_{|D|\leqslant X}' \left| \sum_{1\leqslant n\leqslant Y} \left(\frac{D}{n}\right)\right|^2, \tag{1}$$

where, as also later, $\sum'$ denotes a sum restricted to the set $\mathscr{D}$. By the classical Polya–Vinogradov estimate (shortly: P–V), the sum $S(X, Y)$ is for $X \geqslant 3$, $Y \geqslant 1$ at most $\ll X^2 \log^2 X$. Our result is as follows:

THEOREM 1. *For $X \geqslant 3$, $Y \geqslant 1$ we have uniformly*

$$S(X, Y) \ll XY \log^8 X. \tag{2}$$

In the proof we need only classical arguments: a lemma of I. M. Vinogradov and its corollary (Lemmas 1 and 2), the law of reciprocity for real characters, and the estimate of Polya–Vinogradov.

2. It turns out that the sum over n in (1) may be restricted to fundamental discriminants without essential loss of sharpness of the estimate. Define

$$S_*^+(X, Y) = \sum_{|D| \leqslant X}' \left| \sum_{1 \leqslant n \leqslant Y}^* \left(\frac{D}{n}\right) \right|^2,$$

where $\sum^*$ denotes a summation restricted to fundamental discriminants. Similarly, let $S_*^-(X, Y)$ be defined as above but with n running over $-Y \leqslant n \leqslant -1$.

COROLLARY. *For $X \geqslant 3$, $Y \geqslant 1$ we have uniformly*

$$S_*^+(X, Y), S_*^-(X, Y) \ll XY \log^{10} X. \tag{3}$$

3. As an application of the corollary, we shall prove a mean-value theorem for $h^k(-d)$, with $h(-d)$ the class number of the quadratic field $\mathbf{Q}(\sqrt{-d})$ with the negative discriminant $-d$ and $k \geqslant 1$ a fixed integer.

THEOREM 2. *Let $k \geqslant 1$ be an integer. Then for $X \geqslant 3$ we have*

$$\sum_{1 \leqslant d \leqslant X}^* h^k(-d) = a(k)\, X^{\frac{1}{2}(k+2)}(1 + O(X^{-1/2}(\log X)^{c_1(k)})),$$

where $a(k)$, $c_1(k)$ and the O-constant depend on k only.

Using the Dirichlet formula $h(-d) = \pi^{-1} \mid d \mid^{1/2} L(1, \chi^{(-d)})$, valid for $d > 4$ (see [3], Chap. 5, Sec. 4), Theorem 2 is by partial summation obtained from the following

THEOREM 3. *With k and X as in Theorem 2, we have*

$$\sum_{d \in [e, eX]}^* L^k(1, \chi^{(d)}) = b(k)X + O(X^{1/2}(\log X)^{c_2(k)}),$$

where $e = \pm 1$, and $b(k)$, $c_2(k)$ and the O-constant depend on k only.

NOTES

4. Wolke has in [9] estimated a double sum for quadratic characters, using the "large-sieve" inequality of Linnik–Bombieri–Gallagher. It may be noted that, as an easy consequence of Theorem 1, Lemma 1 of [9] can be improved in the case $y \geqslant x^{1/2+\epsilon}$.

5. Sums closely related to the one considered in Theorem 2 have been investigated in the case $k = 1$ by I. M. Vinogradov [6], [7] and in the case $k \geqslant 2$ by Lavrik [4]; Barban [1], [2]; Wolke [9]; and also by some other writers. These papers concern the k'th power of the class number of the properly primitive positive binary quadratic forms with the negative determinant $-d$. In other words, the forms in question are of the type $ax^2 + 2bxy + cy^2$ with $a > 0$, $(a, 2b, c) = 1$, $-d = b^2 - ac < 0$. An estimate of the same strength as Theorem 2 can be proved for this class number sum.

A LEMMA OF I. M. VINOGRADOV

6. The following well-known lemma is proved in [8], pp. 32–34.

LEMMA 1. *Let α, β, and Δ be real numbers, satisfying*

$$0 < \Delta < \tfrac{1}{2}, \qquad \Delta \leqslant \beta - \alpha \leqslant 1 - \Delta.$$

Then there exists a periodic function $\psi(x)$ with period 1, satisfying

(i) $\psi(x) = 1$ *in the interval* $\alpha + \frac{1}{2}\Delta \leqslant x \leqslant \beta - \frac{1}{2}\Delta$,

(ii) $\psi(x) = 0$ *in the interval* $\beta + \frac{1}{2}\Delta \leqslant x \leqslant 1 + \alpha - \frac{1}{2}\Delta$,

(iii) $0 \leqslant \psi(x) \leqslant 1$ *in the intervals* $\alpha - \frac{1}{2}\Delta \leqslant x \leqslant \alpha + \frac{1}{2}\Delta, \beta - \frac{1}{2}\Delta \leqslant x \leqslant \beta + \frac{1}{2}\Delta$,

(iv) $\psi(x)$ *has the Fourier expansion*

$$\psi(x) = \beta - \alpha + \sum_{m=1}^{\infty} (a_m e^{2\pi i m x} + \bar{a}_m e^{-2\pi i m x}), \tag{4}$$

where

$$a_m = (2m\pi i)^{-1}(e^{-2\pi i m \alpha} - e^{-2\pi i m \beta}) \left\{ \frac{\sin(\frac{1}{2}\pi m \Delta)}{\frac{1}{2}\pi m \Delta} \right\}^2, \tag{5}$$

$$| a_m | \ll \min\{\beta - \alpha, m^{-1}, \Delta^{-2} m^{-3}\}, \tag{6}$$

(v) *if* α *and* Δ *are fixed but* β *variable in the interval* $B = [\alpha + \Delta, \alpha + 1 - \Delta]$, *and the function* $\psi(x)$, *corresponding to a particular* β, *is denoted by* $\psi_\beta(x)$, *then we have*

$$\psi_{\beta'}(x) = \psi_\beta(x + \beta - \beta') \tag{7}$$

for all $\beta, \beta' \in B$, *and for* $\beta' - \min(\beta - \alpha - \frac{1}{2}\Delta, \beta' - \alpha - \frac{1}{2}\Delta) \leqslant x \leqslant \beta' + \min(1 + \alpha - \frac{1}{2}\Delta - \beta, 1 + \alpha - \frac{1}{2}\Delta - \beta')$.

Indeed, take $r = 2$ in Lemma 2 of [8]. The additional assertion (v) follows from the method of construction of the function $\psi(x)$.

For future reference, it is convenient to state some further properties of $\psi(x)$.

LEMMA 2. *Denote by* $\psi(x, u)$ *the function* $\psi(x)$ *of Lemma* 1 *with the following choice of the parameters*:

$$\Delta = X^{-1/2}, \qquad \alpha = \tfrac{1}{2}\Delta, \qquad \beta = Yu^{-1} + \tfrac{1}{2}\Delta = \beta(u), \tag{8}$$

where

$$X^{1/2} \leqslant Y \leqslant \tfrac{1}{4}X, \qquad \tfrac{1}{2}X \leqslant u \leqslant X, \qquad X > 100. \tag{9}$$

Then

$$\frac{\partial \psi(x, u)}{\partial x} \ll \Delta^{-1}. \tag{10}$$

Further,

$$\frac{d\psi\left(\dfrac{n}{u}, u\right)}{du} \ll X^{-1} \tag{11}$$

if

$$1 \leqslant n \leqslant X^{1/2}, \qquad or \qquad Y \leqslant n \leqslant Y + X^{1/2}. \tag{12}$$

Finally, if $a_{m,u}$ *are the Fourier coefficients of the expansion of the function* $\psi(x, u)$, *then*

$$\frac{da_{m,u}}{du} \ll \min\{YX^{-2}, YX^{-1}m^{-2}\}. \tag{13}$$

The estimates (10), (11), *and* (13) *hold uniformly with respect to all parameters and variables.*

Proof. For the proof of (10), note that the expansion (4) can be differentiated termwise, and use (6).

The function $\psi(x, u)$ is, for $0 \leqslant x \leqslant 2\Delta$, independent of u; hence for $1 \leqslant n \leqslant X^{1/2}$ we have by (10), (8), and (9) the estimate (11).

For the remaining case $Y \leqslant n \leqslant Y + X^{1/2}$ of (11), we use (v) of Lemma 1 with $\beta = (Y/u) + \frac{1}{2}\Delta$, $\beta' = [Y/(u+\eta)] + \frac{1}{2}\Delta$. We have

$$\begin{aligned}\frac{d\psi\left(\frac{n}{u}, u\right)}{du} &= \lim_{\eta\to 0} \eta^{-1}\left(\psi\left(\frac{n}{u+\eta}, u+\eta\right) - \psi\left(\frac{n}{u}, u\right)\right) \\ &= \lim_{\eta\to 0} \eta^{-1}\left(\psi\left(\frac{n}{u+\eta} + \frac{Y}{u} - \frac{Y}{u+\eta}, u\right) - \psi\left(\frac{n}{u}, u\right)\right) \\ &\ll \lim_{\eta\to 0} \eta^{-1}\Delta^{-1}(n - Y)\left(\frac{1}{u+\eta} - \frac{1}{u}\right) \ll X^{-1},\end{aligned}$$

and (11) is established.

The estimate (13) is obtained from (5) by a simple calculation.

Proof of Theorem 1

7. The case $Y \geqslant \frac{1}{4}X$ is clear by P–V, and the case $Y \leqslant X^{1/2}$ is also easy, as seen from the following calculation. First,

$$S(X, Y) \leqslant \sum_{\substack{r,s\leqslant Y \\ rs\neq a^2}} \sum_{|D|\leqslant X}' \left(\frac{D}{rs}\right) + \sum_{n^2\leqslant Y^2} \tau(n^2) \sum_{|D|\leqslant X}' 1. \tag{14}$$

To estimate the second double sum, note that $\tau(n^2) \leqslant \tau^2(n)$, and that

$$\sum_{n\leqslant Y} \tau^2(n) \ll Y \log^3 Y,$$

whence the resulting contribution to $S(X, Y)$ is $\ll XY \log^3 Y$.

Modify the first double sum removing the condition (i) from the definition of the set $\mathscr{D}$. This gives rise to an error $\ll Y^2X^{1/2} \leqslant XY$ which can be neglected.

In the resulting double sum, let first rs be fixed and consider the values of the symbol

$$\left(\frac{D}{rs}\right).$$

For the values of D in question, we have

$$\left(\frac{D}{rs}\right) = \chi(D),$$

where χ is a nonprincipal character to a modulus $\leqslant 4rs$, having a conductor different from 4. Hence the D-sum can be expressed by a finite number of character sums with nonprincipal characters to a modulus $\leqslant 16rs$. So, P–V, the double sum in consideration is

$$\ll \sum_{r,s\leqslant Y} (rs)^{1/2} \log Y \ll Y^3 \log X \leqslant XY \log X.$$

Hence Theorem 1 is found to hold in the case $Y \leqslant X^{1/2}$.

8. Now we proceed to the more difficult case $X^{1/2} \leqslant Y \leqslant \frac{1}{4}X$. Since we shall apply Lemma 2 with $u = |D|$, it is convenient to restrict $|D|$ to the interval $[\frac{1}{2}X, X]$, and the sum over this interval being estimated, apply the result with $\frac{1}{2}X$ in place of X, etc. Furthermore, we first restrict D to the sequence of the fundamental discriminants.

The first step is to express the inner sum (e.g., $S^{(D)}$) of $S(X, Y)$ in terms of the function $\psi(x, u)$ of Lemma 2 as follows:

$$\begin{aligned} S^{(D)} &= \sum_{1\leqslant n\leqslant Y} \left(\frac{D}{n}\right) = \sum_{1\leqslant n\leqslant |D|} \psi\left(\frac{n}{|D|}, |D|\right)\left(\frac{D}{n}\right) \\ &\quad + \sum_{1\leqslant n\leqslant X^{1/2}} \left(1 - \psi\left(\frac{n}{|D|}, |D|\right)\right)\left(\frac{D}{n}\right) \\ &\quad - \sum_{Y\leqslant n\leqslant Y+X^{1/2}} \psi\left(\frac{n}{|D|}, |D|\right)\left(\frac{D}{n}\right) \\ &= S_1^{(D)} + S_2^{(D)} + S_3^{(D)}. \end{aligned} \tag{15}$$

Hence, writing S for the part of $S(X, Y)$ in consideration, we have

$$S = \sideset{}{^*}\sum_{\frac{1}{2}X\leqslant |D|\leqslant X} |S^{(D)}|^2 \ll \sum_{i=1}^{3} \sideset{}{^*}\sum_{\frac{1}{2}X\leqslant |D|\leqslant X} |S_i^{(D)}|^2 = S_1 + S_2 + S_3. \tag{16}$$

9. *Treatment of the sum S_1.* Using the representation

$$\psi(x, u) = \beta(u) - \alpha + \sum_{m=1}^{\infty} (a_{m,u}e^{2\pi imx} + \bar{a}_{m,u}e^{-2\pi imx})$$

and the primitivity of the character $\left(\frac{D}{n}\right)$ in (15), we obtain

$$\begin{aligned} S_1^{(D)} &= \tau_D \sum_{m=1}^{\infty} \left(a_{m,|D|}\left(\frac{D}{m}\right) + \bar{a}_{m,|D|}\left(\frac{D}{-m}\right)\right) \\ &= \tau_D \left(\sum_{m\leqslant X^{1/2}} + \sum_{m>X^{1/2}}\right) = S_{11}^{(D)} + S_{12}^{(D)}, \end{aligned}$$

where τ_D is a Gaussian sum. So

$$S_1 \ll \sum_{\frac{1}{2}X \leqslant |D| \leqslant X}^{*} (|S_{11}^{(D)}|^2 + |S_{12}^{(D)}|^2) = S_{11} + S_{12}.$$

Since $|\tau_D|^2 = |D|$, we have further, allowing D now run over $\mathscr{D}$,

$$S_{11} \ll \sum_{\frac{1}{2}X \leqslant |D| \leqslant X}' |D| \left| \sum_{m \leqslant X^{1/2}} a_{m,|D|} \left(\frac{D}{m}\right)\right|^2$$

$$= \sum_{r,s \leqslant X^{1/2}} \sum_{\frac{1}{2}X \leqslant |D| \leqslant X}' |D| \, a_{r,|D|} \bar{a}_{s,|D|} \left(\frac{D}{rs}\right), \tag{17}$$

and analogously for S_{12}.

To estimate the sum S_{11}, we consider separately two cases according to whether rs is a square or not.

(A) *rs a square.* By (6) and (8), the contribution of such pairs (r, s) to the sum on the right of (17) is

$$\ll X^2 \sum_{r,s \leqslant X^{1/2}} q(rs) \min(YX^{-1}, r^{-1}) \min(YX^{-1}, s^{-1}), \tag{18}$$

with $q(n) = 1$ if n is a square, and $q(n) = 0$ otherwise.

Split up the double sum (18) into $\ll \log^2 X$ parts of the type

$$\sum(U, V) = \sum_{\substack{U \leqslant r \leqslant U' \\ V \leqslant s \leqslant V'}},$$

with $U' \leqslant 2U$, $V' \leqslant 2V$. Now by (18) obviously

$$\sum(U, V) \ll X^2 \sum_{UV \leqslant n^2 \leqslant 4UV} \tau(n^2) \min(YX^{-1}, U^{-1}) \min(YX^{-1}, V^{-1}). \tag{19}$$

Considering separately the four cases combined from $U \leqslant XY^{-1}$, $U > XY^{-1}$, and $V \leqslant XY^{-1}$, $V > XY^{-1}$, it is easily seen that in any case

$$\sum(U, V) \ll XY \log^3 X,$$

and so all sums of this kind contribute to S_{11} together $\ll XY \log^5 X$.

(B) *rs not a square.* In this case we estimate the inner sums in (17) by partial summation. To this end, we introduce the functions

$$f_{r,s}(u) = u a_{r,u} \bar{a}_{s,u}, \tag{20}$$

$$g_{r,s}(u) = \sum_{\frac{1}{2}X \leqslant |D| \leqslant u}' \left(\frac{D}{rs}\right). \tag{21}$$

Let $p(n) = 0$ if n is a square, and $p(n) = 1$ otherwise. Then the part in consideration of the double sum in (17) is by a well-known formula equal to

$$\sum_{r,s \leqslant X^{1/2}} p(rs) \left\{ g_{r,s}(X) f_{r,s}(X) - \int_{\frac{1}{2}X}^{X} g_{r,s}(u) f'_{r,s}(u)\, du \right\}. \tag{22}$$

Now note that by (6)

$$f_{r,s}(X) \ll X r^{-1} s^{-1}, \tag{23}$$

by P–V (as in Sec. 7)

$$g_{r,s}(u) \ll (rs)^{1/2} \log X + X^{1/2} \ll X^{1/2} \log X, \tag{24}$$

and finally by (13) and (6)

$$f'_{r,s}(u) \ll YX^{-1}(|a_{r,u}| + |a_{s,u}|) \ll YX^{-1}(r^{-1} + s^{-1}). \tag{25}$$

Using (23)–(25), we obtain for (22) the estimate

$$\ll \sum_{r,s \leqslant X^{1/2}} X^{1/2} \log X (Xr^{-1}s^{-1} + Y(r^{-1} + s^{-1})) \ll XY \log^3 X.$$

We have proved that $S_{11} \ll XY \log^5 X$. The above method works for the sum S_{12} as well. Now r and s run over $r > X^{1/2}$, $s > X^{1/2}$. The Case (A) gives a contribution

$$\ll X^2 \sum_{n^2 > X} \tau(n^2) X^2 n^{-6} \ll X^{3/2} \log^3 X \ll XY \log^3 X.$$

For the Case (B) note that in place of (23)–(25) we now have the estimates $f_{r,s}(X) \ll X^3 r^{-3} s^{-3}$, $g_{r,s}(u) \ll (rs)^{1/2} \log(rs)$, $f'_{r,s}(u) \ll XYr^{-2}s^{-2}(r^{-1} + s^{-1})$, and by partial summation we obtain a contribution $\ll XY \log X$.

To summarize, $S_{12} \ll XY \log^3 X$, hence by the above estimate for S_{11} we have finally

$$S_1 \ll XY \log^5 X. \tag{26}$$

10. *Treatment of the sums S_2 and S_3.* We shall consider in detail the sum S_3 only since the case of S_2 is quite analogous.

By (15) and (16) we have

$$S_3 \leqslant \sum_{Y \leqslant r,s \leqslant Y + X^{1/2}} \sideset{}{'}\sum_{\frac{1}{2}X \leqslant |D| \leqslant X} \psi\left(\frac{r}{|D|}, |D|\right) \psi\left(\frac{s}{|D|}, |D|\right) \left(\frac{D}{rs}\right).$$

Again distinguish two cases according to whether rs is a square or not. The squares give a contribution $\ll X^{3/2} \log^3 X \ll XY \log^3 X$, as easily calculated.

The remaining sum is by partial summation equal to

$$\sum_{Y\leqslant r,s\leqslant Y+X^{1/2}} p(rs)\Big\{\psi\Big(\frac{r}{X}, X\Big)\,\psi\Big(\frac{s}{X}, X\Big)\,g_{r,s}(X) - \int_{\frac{1}{2}X}^{X} g_{r,s}(u)\,\frac{d}{du}\Big(\psi\Big(\frac{r}{u}, u\Big)\,\psi\Big(\frac{s}{u}, u\Big)\Big)\,du\Big\}.$$

By (11) and P–V, this is

$$\ll \sum_{Y\leqslant r,s\leqslant Y+X^{1/2}} (rs)^{1/2}\log X \ll XY\log X.$$

Hence, $S_3 \ll XY\log^3 X$, and similarly for S_2. So, in view of (26) and (16), we have $S \ll XY\log^5 X$, and hence

$$\sideset{}{^*}\sum_{|D|\leqslant X}\Big|\sum_{1\leqslant n\leqslant Y}\Big(\frac{D}{n}\Big)\Big|^2 \ll XY\log^6 X. \tag{27}$$

11. *Completion of the proof.* In view of (27), we only have to estimate the contribution of the nonfundamental values of D to the sum $S(X, Y)$.

In general, if $\left(\frac{d}{n}\right)$ is the primitive character, equivalent to $\left(\frac{D}{n}\right)$, and $D = dd_1$, we have

$$\Big|\sum_{1\leqslant n\leqslant Y}\Big(\frac{D}{n}\Big)\Big|^2 = \Big|\sum_{\substack{1\leqslant n\leqslant Y\\(n,d_1)=1}}\Big(\frac{d}{n}\Big)\Big|^2 = \Big|\sum_{\delta/d_1}\mu(\delta)\Big(\frac{d}{\delta}\Big)\sum_{1\leqslant n\leqslant Y\delta^{-1}}\Big(\frac{d}{n}\Big)\Big|^2$$
$$\leqslant \tau(d_1)\sum_{\delta/d_1}\Big|\sum_{1\leqslant n\leqslant Y\delta^{-1}}\Big(\frac{d}{n}\Big)\Big|^2.$$

Hence for fixed values of d and δ with $|d|\,\delta \leqslant X$ we have to $S(X, Y)$ a contribution

$$\leqslant \Big|\sum_{1\leqslant n\leqslant Y\delta^{-1}}\Big(\frac{d}{n}\Big)\Big|^2\sum_{\substack{d_1\leqslant X|d|^{-1}\\ \delta/d_1}}\tau(d_1) \ll \Big|\sum_{1\leqslant n\leqslant Y\delta^{-1}}\Big(\frac{d}{n}\Big)\Big|^2\tau(\delta)\,X\,|d|^{-1}\,\delta^{-1}\log X.$$

Now a summation over $|d| \leqslant X\delta^{-1}$ for a fixed δ gives by (27) a contribution $\ll XY\delta^{-2}\tau(\delta)\log^8 X$. Finally, a summation over δ accomplishes the proof of Theorem 1.

Proof of the Corollary

12. Consider $S_*^+(X, Y)$. Let generally in a sum $\sum_n''$ the number n run over the integers $n = \nu a^2$, where a is odd and ν a fundamental discriminant. Then obviously

$$\Big|\sum_{1\leqslant n\leqslant Y}^{*}\Big(\frac{D}{n}\Big)\Big| = \Big|\sum_{1\leqslant n\leqslant Y}''\Big(\sum_{\substack{\delta^2\mid n\\ 2\nmid\delta}}\mu(\delta)\Big)\Big(\frac{D}{n}\Big)\Big| \leqslant \sum_{\substack{\delta\leqslant Y^{1/2}\\ 2\nmid\delta}}\Big|\sum_{\substack{1\leqslant n\leqslant Y\\ \delta^2\mid n}}''\Big(\frac{D}{n}\Big)\Big|$$

$$\leqslant \sum_{\substack{\delta\leqslant Y^{1/2}\\ 2\nmid\delta}}\big(\Sigma_1^{(D,\delta)} + \Sigma_2^{(D,\delta)}\big), \tag{28}$$

where

$$\Sigma_1^{(D,\delta)} = \Big|\sum_{\substack{1\leqslant n\leqslant Y\\ 4\delta^2\mid n}}''\Big(\frac{D}{n}\Big)\Big|,$$

and $\Sigma_2^{(D,\delta)}$ a similar sum with the conditions $n \equiv 1 \pmod 4$, $\delta^2 \mid n$ instead of the condition $4\delta^2 \mid n$.

In $\Sigma_1^{(D,\delta)}$, n runs over the numbers $n = 4\delta^2 k$, $k \equiv 2, 3 \pmod 4$ and in $\Sigma_2^{(D,\delta)}$ over the numbers $n = \delta^2 k$, $k \equiv 1 \pmod 4$. Hence it is easily seen that

$$\Sigma_1^{(D,\delta)} \leqslant \Big|\sum_{k\leqslant Y/4\delta^2}\Big(\frac{D}{k}\Big)\Big| + \Big|\sum_{k\leqslant Y/16\delta^2}\Big(\frac{D}{k}\Big)\Big|$$

$$+ \Big|\sum_{k\leqslant Y/4\delta^2}\chi_4(k)(1+\chi_4(k))\Big(\frac{D}{k}\Big)\Big|, \tag{29}$$

$$\Sigma_2^{(D,\delta)} \leqslant \Big|\sum_{k\leqslant Y/\delta^2}\chi_4(k)(1+\chi_4(k))\Big(\frac{D}{k}\Big)\Big|, \tag{30}$$

where χ_4 is the primitive character (mod 4).

From (28)–(30), we see that the proof of the estimate of $S_*^+(X, Y)$ is reduced to the proof of the estimate

$$\sum_{|D|\leqslant X}'\Big(\sum_{\substack{\delta\leqslant Y^{1/2}\\ 2\nmid\delta}}\Big|\sum_{n\leqslant Y\delta^{-2}}\Big(\frac{D}{n}\Big)\Big|\Big)^2 \ll XY\log^{10} X.$$

But this is easily verified by Schwarz's inequality and Theorem 1. Since the case of $S_*^-(X, Y)$ is quite analogous, the Corollary is verified.

Proof of Theorem 3

13. Starting from the formula

$$L(1, \chi^{(d)}) = \sum_{n \leqslant X} \frac{\left(\frac{d}{n}\right)}{n} + O(X^{-1/2} \log X),$$

valid for all d with $|d| \leqslant X$, we have

$$L^k(1, \chi^{(d)}) = \sum_{n \leqslant X^k} \frac{\tau_k'(n) \left(\frac{d}{n}\right)}{n} + O_k(X^{-1/2} \log^k X),$$

where $\tau_k'(n) = \tau_k(n)$ for $n \leqslant X$, and $\tau_k'(n) \leqslant \tau_n(n)$ for $n > X$. So, on summation over d, we obtain

$$\sum_{d \in [e, eX]}^{*} L^k(1, \chi^{(d)}) = \sum_{n^2 \leqslant X} \frac{\tau_k(n^2)}{n^2} \sum_{\substack{d \in [e, eX] \\ (d,n)=1}}^{*} 1 + O\left(\sum_{n^2 > X} \frac{\tau_k(n^2)}{n^2} X\right)$$
$$+ O_k(X^{1/2} \log^k X) + O\left(\sum_{\substack{n \leqslant X^k \\ n \neq a^2}} \frac{\tau_k(n) |S(n)|}{n}\right), \tag{31}$$

where

$$S(n) = \sum_{d \in [e, eX]}^{*} \left(\frac{d}{n}\right).$$

Consider the main term first. It is easily verified that

$$\sum_{\substack{d \in [e, eX] \\ (d,n)=1}}^{*} 1 = A(n) X + O(X^{1/2} \tau(n))$$

on expressing the sum by the Mobius function. Hence the main term in (31) is

$$b(k) X + O_k(X^{1/2}),$$

where

$$b(k) = \sum_{n=1}^{\infty} A(n)\, \tau_k(n^2)\, n^{-2}.$$

To prove Theorem 3, we have to prove an estimate for the last term in (31), and the result

$$\sum_{\substack{n \leqslant X^k \\ n \neq a^2}} \frac{|S(n)|}{n} = O_k(X^{1/2} \log^6 X) \tag{32}$$

would suffice. But by the reciprocity law for real characters, we can write the sum $S(n)$ as a sum of the type considered in $S_*^+(X, Y)$ and $S_*^-(X, Y)$. Hence, by Schwarz's inequality and the Corollary, the estimates (32) is established, and the proof of Theorem 3 is complete.

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